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Kink Chains from Instantons on a Torus

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Abstract

We describe how the procedure of calculating approximate solitons from instanton holonomies may be extended to the case of soliton crystals. It is shown how sine-Gordon kink chains may be obtained from $\mathbb{CP}^1$ instantons on T^2. These kink chains turn out to be remarkably accurate approximations to the true solutions. Some remarks on the relevance of this work to Skyrme crystals are also made.

1 Introduction

The construction of Skyrme fields from instantons was introduced a few years ago by Atiyah and Manton [1]. Not only has it has proved useful in understanding the solitons of the Skyrme model, but the general procedure of obtaining approximate solitons from higher dimensional instantons has been shown to apply to several other systems. However, for all the cases studied so far the mean soliton density $\bar{\rho} = (number\ of\ solitons)/(volume\ of\ space)$ is zero. Of course, for soliton theories considered in non-compact space this condition is a requirement for the configuration to have finite energy. However, there are many situations of physical interest for which $\bar{\rho} \neq 0$. Two examples of relevance to the work in this paper are kinks of the sine-Gordon model on a circle, which describe fluxons in long (but finite) Josephson junctions [2], and Skyrme crystals, which are relevant for describing regions of high baryon density such as inside a neutron star [3]. By soliton crystal we mean a configuration which is periodic in space and contains a finite (non-zero) number of solitons in a unit cell. By definition such a configuration (excluding the case of infinite period) will have non-zero $\bar{\rho}$, and both the above mentioned examples may be viewed as soliton crystals.

In this paper we shall concentrate on the case of the periodic sine-Gordon equation, where the soliton crystal is a kink chain. It will be shown in detail how an approximate kink chain may be constructed from an instanton on the torus. Since the exact kink chain solution is known we are also able to make a detailed study of the accuracy of the instanton generated approximation. In the limit in which the kink period tends to zero the instanton approximation becomes exact. In the limit of an infinite kink period there are two cases to consider, depending on whether the length of the torus in the euclidean time direction is infinite or finite. In the first case we reproduce the already known approximation obtained from instantons on $\mathbb{R}^2$, and in the second case we find a new, and substantially more accurate, approximation which decays exponentially, in agreement with the true kink solution.

We view this work as a simple lower dimensional analogue of the problem for Skyrme crystals, where an explicit exact solution is not known and the instanton method may prove useful. Some remarks are made on issues raised by the analysis here that are also relevant to the Skyrme crystal.

2 Kink Chains

In this section we shall briefly review some results for the periodic sine-Gordon equation in one space dimension. The real sine-Gordon field $\phi(x,t)$ is taken to be periodic (mod 2π) in the space variable x with period L *ie*

$$\phi(x+L,t) = \phi(x,t) + 2\pi n \tag{2.1}$$

for some $n \in \mathbb{Z}$. The integrability of the model allows the construction of explicit multi-soliton wave train solutions using, for example, the inverse scattering transform [4]. Fortunately we shall not need these complicated solutions, involving vector Θ-functions, since we wish to consider the case of a static kink crystal. This is a time independent kink chain, satisfying (2.1) with $n = 1$, for which the solution may be given simply in terms of a Jacobi elliptic function. Explicitly, we consider a segment of the kink chain, $x \in [0, L]$, with the boundary conditions $\phi(x=0) = 0$ and $\phi(x=L) = 2\pi$. Then this segment contains precisely one kink (or soliton). Since we are considering only static configurations the energy per soliton is given by

$$\mathcal{E} = \int_0^L \frac{1}{16}(\partial_x \phi)^2 + \frac{1}{8}(1 - \cos\phi)\ dx. \tag{2.2}$$

By completing the square in the above energy density we obtain, in the usual way, a Bogomolny bound $\mathcal{E} \geq 1$ with equality if and only if

$$\partial_x \phi = 2\sin\frac{\phi}{2}. \tag{2.3}$$

The static sine-Gordon equation determined by (2.2) is

$$\partial_x^2 \phi = \sin\phi \tag{2.4}$$

which can be integrated once to give

$$\partial_x \phi = 2\sqrt{\sin^2(\frac{\phi}{2}) + \frac{1-\widetilde{m}}{\widetilde{m}}} \tag{2.5}$$

where $\widetilde{m} \in (0, 1]$ is a constant of integration which is related to the period L in a way given below. Comparing (2.3) with (2.5) we see that the Bogomolny

bound is attained only if $\widetilde{m}=1$, which corresponds (see below) to the limit $L\to\infty$. For $\widetilde{m}\neq 1$ we see that the Bogomolny bound is exceeded. A simple change of variable allows the integration of (2.5) to give the kink chain solution

$$\phi = 2\arcsin(\mathrm{cn}(\frac{x-\frac{L}{2}}{\sqrt{\widetilde{m}}}|\widetilde{m})). \tag{2.6}$$

Here the kink is situated in the centre of the interval and we have used the standard notation (see for example [7]) that $\mathrm{cn}(x|m)$ denotes the appropriate Jacobi elliptic function with argument x and parameter m. The period of the elliptic function must be equal to L which gives the relation between $\widetilde{m}$ and L

$$L = 2\sqrt{\widetilde{m}}\widetilde{K} \tag{2.7}$$

where $\widetilde{K}$ denotes the complete elliptic integral of the first kind corresponding to the parameter $\widetilde{m}$ *ie*

$$\widetilde{K} = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-\widetilde{m}\sin^2\theta}}. \tag{2.8}$$

Substituting the solution (2.6) into the energy formula (2.2) gives the energy to be

$$\mathcal{E} = \frac{1}{\sqrt{\widetilde{m}}}(\widetilde{E} - \frac{1}{2}(1-\widetilde{m})\widetilde{K}) \tag{2.9}$$

where $\widetilde{E}$ denotes the complete elliptic integral of the second kind with parameter $\widetilde{m}$. The energy tends to the Bogomolny bound $\mathcal{E}=1$ as $\widetilde{m}\to 1$, and is strictly monotonic increasing as $\widetilde{m}$ decreases, indicating that there are repulsive forces between kinks.

Let us mention the two extreme limits of L. The first is the small period limit $L\to 0$, which is also the weak coupling limit of the theory since the gradient energy will dominate the potential term. Hence the asymptotic limit of the solution will be the linear function

$$\phi \sim \frac{2\pi x}{L}. \tag{2.10}$$

The second is the infinite period limit $L\to\infty$, when the solution reduces to the standard expression for the sine-Gordon kink on $\mathbb{R}$. In this situation it is convenient to position the kink at the origin $x=0$, which is simply achieved

by shifting the x coordinate by $-\frac{L}{2}$. Taking the limit $L \to \infty$ the solution (2.6) takes the familiar form

$$\phi = 4 \arctan(\mathrm{e}^x). \tag{2.11}$$

3 $\mathbf{CP}^1$ Instantons on $\mathbf{T}^2$

In this section we study the $\mathbb{CP}^1$ σ-model defined on the torus with euclidean metric. We use the gauge field formulation, defined in terms of a 2-component column vector V, which is a function of the spacetime coordinates, $x^\mu = (x^1, x^2) = (x, y)$, with metric $\delta_{\mu\nu} = \mathrm{diag}(1, 1)$ on T^2. The vector is constrained to satisfy the normalization condition

$$V^\dagger V = 1. \tag{3.1}$$

The action has a U(1) gauge symmetry and is given by

$$S = \int_{T^2} \mathrm{Tr}\ (D_\mu V)^\dagger (D^\mu V)\ d^2x \tag{3.2}$$

where Tr denotes trace and D_μ are the covariant derivatives

$$D_\mu = \partial_\mu - A_\mu \tag{3.3}$$

with the composite gauge fields being purely imaginary and defined by

$$A_\mu = V^\dagger \partial_\mu V. \tag{3.4}$$

It is convenient to also consider a gauge fixed formulation of the model, obtained by introducing the parametrization

$$V = \frac{1}{\sqrt{1 + |W|^2}} \begin{pmatrix} 1 \\ W \end{pmatrix} \tag{3.5}$$

where $W \in \mathbb{C} \cup \{\infty\}$. In terms of this formulation the action (3.2) becomes

$$S = \int_{T^2} \frac{\partial_\mu W \partial^\mu \bar{W}}{(1 + |W|^2)^2}\ d^2x \tag{3.6}$$

and the Euler-Lagrange equation derived from (3.6) is

$$\partial_\mu \partial^\mu W = \frac{2\bar{W}\partial_\mu W \partial^\mu W}{(1+|W|^2)}. \tag{3.7}$$

We are interested in finite action solutions of this equation; which in the language of differential geometry are harmonic maps from T^2 to S^2, since $\mathbb{CP}^1$ is isomorphic to the Riemann sphere. Such maps have an associated degree as described below. Let $\zeta, \bar{\zeta}$ be coordinates on $\mathbb{CP}^1$, and using the Fubini-Study metric construct the standard volume 2-form

$$\omega = \frac{d\zeta \wedge d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}. \tag{3.8}$$

Then for a given map, $W: \ T^2 \to \mathbb{CP}^1$, the degree ($degW$) is determined by pulling back the volume 2-form (3.8) to T^2 and normalizing by the total volume of $\mathbb{CP}^1$. Explicitly,

$$\int_{T^2} W^*\omega = (degW) \int_{\mathrm{CP}^1} \omega. \tag{3.9}$$

Introducing the complex coordinate $z = x + iy$ on the torus and using (3.8) we obtain

$$degW = \frac{1}{\pi} \int_{T^2} \frac{(|\partial_z W|^2 - |\partial_{\bar{z}} W|^2)}{(1+|W|^2)^2} \ dz d\bar{z}. \tag{3.10}$$

We shall denote the degree $degW$ by the integer N, which in physics is known as the topological charge or instanton number. Returning to the gauge formulation it is easy to see that this integer is precisely the (suitably normalized) magnetic charge

$$N = \frac{1}{i2\pi} \int_{T^2} F_{xy} \ d^2x \tag{3.11}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the abelian field strength.

We now recall some results from differential geometry concerning the harmonic maps we are interested in. The first theorem states that [5]

Any harmonic map from a compact Riemann surface of genus g to the two-sphere is holomorphic, provided its degree is greater than or equal to g.

Hence all the solutions we are interested in with positive instanton number N are given by holomorphic maps. (Those with $N < 0$ are given by anti-holomorphic maps). It is easily seen that taking W to be a holomorphic function of z solves the equation (3.7). These are the so called instanton solutions. It is clear that no solution can exist with $N = 1$ since this would imply that T^2 and S^2 were diffeomorphic. The question of which solutions exist is answered by the following theorem [6]

Holomorphic representatives for all maps from a Riemann surface of genus g to the two-sphere exist for degree greater than g.

Applying the above theorems to our case of interest we conclude that all finite action solutions are instantons, which exist if and only if $N \neq \pm 1$.

Let us now make a brief comment on the relevance of zeros and poles of W. In the case of $\mathbb{CP}^1$ instantons on $\mathbb{R}^2$, finite energy requires that the field tends to the same value at spatial infinity irrespective of the direction of approach *ie* there exists a constant vacuum value W_{vac} such that $W(z) \to W_{vac}$ as $|z| \to \infty$. Then $\widetilde{W}_{vac} = -\bar{W}_{vac}^{-1}$, which is the point on the target manifold 2-sphere antipodal to W_{vac}, is the value associated with the W field at the position of an instanton. That is, an instanton has position $z = z_0$ if $W(z_0) = \widetilde{W}_{vac}$. Usually the vacuum value is taken to be $W_{vac} = 0$ (or ∞) then $\widetilde{W}_{vac} = \infty$ (or 0) so that poles (or zeros) of W are important since they are associated with the location of instantons. For the instantons considered in this paper there is no concept of a vacuum value since T^2 is compact. However, we shall see that the poles and zeros of W will still be important in interpreting the location of an instanton.

We have already given a definition (3.9) of the degree of a map from T^2 to S^2, but an alternative (and of course equivalent) definition is also useful. The degree of the map is equal to the number of preimages (counted with the sign of the Jacobian of the mapping) of a given generic image point in S^2. As we have seen the simplest case corresponds to $N = 2$ ($N = 0$ is trivial and no solutions exist for $N = 1$). So in searching for a 2-instanton solution we require a holomorphic elliptic function with two simple poles in a unit cell. Thus Jacobi elliptic functions provide examples of 2-instanton solutions. It is such a solution which we shall use in this paper.

The notation and results on elliptic functions that we shall use may be found in [7]. Let $m \in [0,1]$ be the elliptic function parameter with $m' = 1 - m$ the complementary parameter. Denote by K and K' the complete elliptic integrals of the first kind corresponding to m and m' respectively. As mentioned above the locations of the zeros and poles of an instanton solution will be of interest. For the Jacobi elliptic functions there is a simple diagrammatic representation of these points, as we now describe. Let Ω denote the set of letters $\{s, c, n, d\}$. Arrange the elements of Ω in a lattice in the complex z-plane (as shown below). Here the s at the bottom left of the lattice is placed at the position $z = x + iy = 0$ and the lattice spacing in the x and y directions is K and K' respectively. If p and q are any two distinct elements of Ω, then the Jacobi elliptic function $pq(z|m)$ has a simple zero at the z value occupied by a p and a simple pole at the z value occupied by a q.

$$\begin{array}{ccccc} s & c & s & c & s \\ n & d & n & d & n \\ s & c & s & c & s \\ n & d & n & d & n \\ s & c & s & c & s \end{array}$$

Fig 1: Schematic representation of the zeros and poles of Jacobi elliptic functions.

The symmetries that we shall later require of the instanton solution (see section 4), together with some simplifications made for ease of analysis, single out the function $dn(z|m)$ as the instanton we require for our purposes. This function has real period $2K$ and imaginary period $4K'$, (*ie* $dn(z|m) = dn(z + 2K|m) = dn(z + 4iK'|m)$) so that the grid covering the torus corresponds to the first three columns in Fig 1. From our second definition of the topological charge it is clear that $dn(z|m)$ describes a 2-instanton, since the letter d occurs twice in the first 3 columns of Fig 1. In fact we wish to scale the torus so that it corresponds to the rectangle $(x, y) \in [0, L] \times [0, \tau]$, with periodic boundary conditions. Explicitly, we take the $\mathbb{CP}^1$ field to be given by

$$W = a\ dn(\frac{2Kz}{L}|m) \tag{3.12}$$

where m is related to τ by $\tau = \frac{2K'L}{K}$. Here a is a positive parameter, which we shall now show determines how localized the instanton is around a zero of W; although it is more complicated than a simple instanton scale or width.

From Fig 1. it is clear that W has zeros at $z = \frac{1}{2}L + i\frac{1}{4}\tau,\ \frac{1}{2}L + i\frac{3}{4}\tau$ and poles at $z = i\frac{1}{4}\tau,\ i\frac{3}{4}\tau$. Let us denote the topological charge density by Q; that is Q is the integrand in the expression (3.10), so that the integral of Q over T^2 gives N. Then for the solution (3.12) we find

$$Q(x,y) = Q(z = x + iy) = \frac{4K^2a^2m^2}{\pi L^2} \frac{|sn(\frac{2Kz}{L}|m)cn(\frac{2Kz}{L}|m)|^2}{(1 + a^2|dn(\frac{2Kz}{L}|m)|^2)^2}. \tag{3.13}$$

Note that although W is periodic on the torus $[0, L] \times [0, \tau]$, the charge density Q is in fact periodic on the half torus $[0, L] \times [0, \frac{1}{2}\tau]$. Hence, even though no $N = 1$ instanton solutions exist, we may at least think of each half of the torus as 'containing one instanton'. We can therefore restrict the following analysis to one zero and one pole of W.

At the zero of W given by $z = \frac{1}{2}L + i\frac{1}{4}\tau$, the topological charge density (3.13) is

$$Q_{\text{zero}} = \frac{4m'a^2K^2}{\pi L^2} \tag{3.14}$$

whereas at the pole $z = i\frac{1}{4}\tau$ it is

$$Q_{\text{pole}} = \frac{4K^2}{\pi L^2 a^2}. \tag{3.15}$$

Therefore we have that

$$\frac{Q_{\text{zero}}}{Q_{\text{pole}}} = m'a^4 \tag{3.16}$$

demonstrating that a determines the extent to which the instanton is localized around the zeros or poles of W. It is easy to confirm this simple picture by plotting (3.13) for various values of a. This shows that indeed for large a the charge density is concentrated in regions around the zeros of W and for small a it is concentrated in regions around the poles of W. For intermediate values $a \sim 1$ the charge density is not well localized around merely zeros or poles but spreads out along lines joining zeros to poles.

4 Instanton Holonomies

The idea of obtaining approximate solitons from instantons was introduced by Atiyah and Manton [1] in the context of the Skyrme model. They showed that calculating the holonomy of an SU(2) Yang-Mills instanton in $\mathbb{R}^4$, along a line parallel to the euclidean time axis, produces a Skyrme field in $\mathbb{R}^3$ with baryon number (topological charge) equal to the instanton number of the original Yang-Mills instanton. Moreover, such Skyrme fields are very good approximations to the soliton solutions (Skyrmions) of the theory. For example, with an appropriate choice of instanton scale the energy of an instanton generated 1-soliton is only 1% higher than that of the (numerically) known solution.

Of relevance to the work in this paper is a lower dimensional analogue of this result, that was introduced by the author in [8]. There it was shown that calculating the holonomy of a $\mathbb{CP}^1$ instanton in $\mathbb{R}^2$, along a line parallel to the euclidean time axis, produces a sine-Gordon kink field in $\mathbb{R}$ with soliton number equal to the instanton number of the original $\mathbb{CP}^1$ instanton. The fundamental relation is

$$\phi(x) = -i \int_{-\infty}^{+\infty} A_y(x,y)\ dy\ + N(\mathrm{mod}2)\pi \tag{4.1}$$

which constructs an N-kink sine-Gordon field ϕ from the euclidean time component A_y of a $\mathbb{CP}^1$ N-instanton. Here we mention only the $N=1$ result, since this will appear later as a limiting case of our periodic construction. From a 1-instanton located at the origin with scale λ the formula (4.1) gives

$$\phi = \pi\left(1 + \frac{\lambda x}{\sqrt{1+\lambda^2 x^2}}\right). \tag{4.2}$$

For some λ this is meant to approximate the exact 1-kink (2.11). The energy of (4.2) is minimized at $\lambda = 0.695$ where it takes the value $\mathcal{E} = 1.010$ *ie* only 1% higher than the exact solution. By closing the contour of integration in (4.1) with a large semicircle at infinity and using Stokes' theorem one can see [9] that ϕ may also be expressed as an area integral of the topological charge density. Explicitly,

$$\phi(x) = 2\pi \int_{-\infty}^{x} dx' \int_{-\infty}^{+\infty} dy\ Q(x',y) \tag{4.3}$$

where $Q(x,y)$ is the topological charge density of the $\mathbb{CP}^1$ instanton in $\mathbb{R}^2$.

We now describe how a periodic analogue of this procedure exists. We construct approximations to the kink chain solutions of section 2 using the instanton on T^2 described in section 3. We use the notation set up in these earlier sections.

It will be convenient to begin from a charge density area integral, of the form (4.3). On T^2 this becomes

$$\phi(x) = \pi \int_0^x dx' \int_0^\tau dy\ Q(x',y) \tag{4.4}$$

where Q is given by equation (3.13) for the instanton we are considering. It is immediately clear that by construction ϕ satisfies the periodic kink chain boundary conditions $\phi(0) = 0$ and $\phi(L) = 2\pi$. Now let us convert this formula to a holonomy-type expression, in order to tie up with the general theory and for ease of calculation later. We need at least four coordinate patches in order to cover T^2. However, the kink field has the symmetry that if $\frac{1}{2}L \leq x \leq L$, then $\phi(x) = 2\pi - \phi(L-x)$. This is one of the symmetries that determined the choice of the W function (3.12); *ie* we require that the generated kink also has this symmetry. This means that we need only calculate ϕ on the half interval $0 \leq x \leq \frac{1}{2}L$. In the required region of the torus $(x,y) \in [0, \frac{1}{2}L] \times [0,\tau]$ we require only two patches. We take these to be $U = \{(x,y) \in (\frac{1}{4}L - \epsilon, \frac{1}{2}L] \times [0,\tau]\}$ and $\hat{U} = \{(x,y) \in [0, \frac{1}{4}L + \epsilon) \times [0,\tau]\}$, where ϵ is a small parameter with $0 < \epsilon < \frac{1}{4}L$. We can in practice reduce the overlap between U and $\hat{U}$ to a minimum by considering their closure by setting $\epsilon = 0$. Note that W (given by (3.12)) has no singularities in U and no zeros in $\hat{U}$. In order to consider holonomies we need to revert to the gauge theory formulation. Recall that in this formulation $Q = \frac{F_{xy}}{i2\pi}$ is of course gauge invariant. The fact that the integral of Q over the whole torus is non-zero represents an obstruction to finding a periodic non-singular gauge potential A_μ over all T^2. We can however find gauge potentials A_μ and $\hat{A}_\mu$ which are non-singular when restricted to the patches U and $\hat{U}$ respectively. Explicitly we have

$$A_\mu = -i\mathrm{Re}\left(\frac{\bar{W}\partial_\mu W}{1+|W|^2}\right) \tag{4.5}$$

and

$$\hat{A}_\mu = -A_\mu |W|^{-2} \tag{4.6}$$

where Re denotes the real part. On the overlap region between U and $\hat{U}$ these two gauge potentials are related by

$$\hat{A}_\mu = A_\mu + i\partial_\mu \alpha \tag{4.7}$$

where

$$\alpha(x,y) = \mathrm{Im}(\log W) \tag{4.8}$$

and Im denotes the imaginary part. Substituting these expressions into (4.4) in the various patches allows the integration over x' to be performed to give the holonomy result

$$\phi(x) = \begin{cases} -i\frac{1}{2}\int_0^\tau (\hat{A}_y(x,y) - \hat{A}_y(0,y))\, dy & \text{if } \ 0 \le x \le \frac{1}{4}L \\ \\ \frac{1}{2}(-i\int_0^\tau (A_y(x,y) - A_y(0,y))\, dy \ + \alpha(\frac{1}{2}L,\tau) - \alpha(\frac{1}{2}L,0)) & \text{if } \frac{1}{4}L < x \le \frac{1}{2}L \end{cases} \tag{4.9}$$

Substituting the explicit expression (3.12) for W in the above and making a scale change of variable for y gives the following final form for ϕ. Introducing the notation

$$\begin{aligned} \mathcal{D} &= dn(\frac{2Kx}{L}|m) \\ \mathcal{S} &= sn(y|m') \\ \Delta &= (1-\mathcal{D}^2)(\mathcal{D}^2 - m') \\ \Xi &= (1-m'\mathcal{S}^4)((1+a^2\mathcal{D}^2) - \mathcal{S}^2(\mathcal{D}^2 + m'a^2))^{-1} \end{aligned}$$

we have the rather complicated expression

$$\phi = \begin{cases} \mathcal{D}\sqrt{\Delta}\int_0^{2K'} \Xi(\mathcal{D}^2 - m'\mathcal{S}^2)^{-1}\, dy & \text{if } 0 \le x \le \frac{1}{4}L \\ \\ \pi - a^2\mathcal{D}\sqrt{\Delta}\int_0^{2K'} \Xi(1 - \mathcal{S}^2\mathcal{D}^2)^{-1}\, dy & \text{if } \frac{1}{4}L < x \le \frac{1}{2}L. \end{cases} \tag{4.10}$$

Remarkably, the above two parameter (m and a) kink field provides an excellent approximation to the kink chain solution for all L.

The two extreme cases $L \to 0$ and $L \to \infty$ can be treated analytically. First we consider the simplest case of $L \to 0$.

In section 2 we discussed the small period limit of the exact solution and stated that in this limit the solution becomes linear (2.10). It is not at all obvious that a linear function limit exists for our instanton generated kink. However, we shall now show that indeed such a limit exists and emerges as we take the limit $m \to 0$ in our construction. To discuss this case it is more convenient to work with the integrated charge density formulation (4.4). In order to produce a field ϕ which depends linearly on x we see that we require the instanton topological charge density to resemble some kind of extended plane wave. With this in mind we see from (3.16) that for $m' \sim 1$ we need to set $a = 1$ to obtain $Q_{zero} = Q_{pole}$. This is a requirement if we wish to obtain a plane wave which passes through the zeros and poles of W. We now make use of the following leading order behaviour of the elliptic functions and integrals as $m \to 0$

$$\begin{aligned}
sn(z|m) &\sim \sin(z) \\
cn(z|m) &\sim \cos(z) \\
dn(z|m) &\sim 1 \\
K &\sim \frac{\pi}{2} \\
K' &\sim \frac{1}{2}\log(\frac{16}{m}).
\end{aligned}$$

Using these results in (4.4) we find the leading order relation for ϕ is

$$\begin{aligned}
\phi &\sim \frac{\pi^2 m^2}{16L^2}\int_0^x dx' \int_0^{\frac{L}{\pi}\log(\frac{16}{m})} (\cos^2(\frac{\pi x'}{L})\sinh^2(\frac{\pi y}{L}) + \sin^2(\frac{\pi x'}{L})\cosh^2(\frac{\pi y}{L}))\; dy \\
&\sim \frac{\pi^2 m^2 x}{64L^2}\int_0^{\frac{L}{\pi}\log(\frac{16}{m})} \exp(\frac{2\pi y}{L})\; dy \\
&\sim \frac{2\pi x}{L}.
\end{aligned} \tag{4.11}$$

Hence we have shown that in the limit in which the period L tends to zero the instanton generated kink approximation becomes exact.

We now consider the limit of infinite period $L \to \infty$. First of all it is more convenient to centre the soliton at $x = 0$, by shifting the x varaiable

$x \to x - \frac{L}{2}$. It turns out that the best parameter choice (in the sense of minimizing the energy of the approximate kink) is to take $m \to 1$ and $a \to \infty$ as $L \to \infty$. Explicitly, these limits are approached with the asymptotic forms

$$\begin{aligned} m &\sim 1 - 16\exp(-bL) \\ a &\sim \frac{\sqrt{\beta}}{4}\exp(\frac{1}{2}bL) \end{aligned} \tag{4.12}$$

where b and β are arbitrary positive constants. Note that for these asymptotic forms we have that

$$\frac{Q_{zero}}{Q_{pole}} = m'a^4 \sim \frac{\beta^2}{16}\exp(bL) \tag{4.13}$$

so that the topological charge density is localized around the zeros of W; as it should be to obtain a kink which decays at spatial infinity in the infinite period case. We now use the leading order behaviour of the elliptic functions and integrals as $m \to 1$

$$\begin{aligned} dn(x|m) &\sim \operatorname{sech}(x) \\ sn(y|m') &\sim \sin(y) \\ K &\sim \frac{1}{2}\log(\frac{16}{m'}) \\ K' &\sim \frac{\pi}{2}. \end{aligned}$$

After substituting these expressions into the holonomy formula (4.10) the integral simplifies sufficiently to be calculated explicitly. The result is

$$\phi(x) = \pi(1 + \frac{\beta \sinh bx \cosh bx}{\sqrt{(1+\beta\cosh^2 bx)(1+\beta\sinh^2 bx)}}). \tag{4.14}$$

Calculating the period of the torus in the euclidean time direction we find it is simply related to the parameter b via $\tau = 2\pi b^{-1}$. So the torus has finite extent in the euclidean time direction if and only if b is non-zero. One expects that the approximate kink (4.2) obtained from instantons in $\mathbb{R}^2$ should appear from the periodic construction if the torus is taken to have infinite period in both directions. By the above remark the limit $\tau \to \infty$ corresponds to the limit $b \to 0$. From (4.14) it can be seen that to obtain a

well defined kink field in this limit requires $\beta \to \infty$. If we take both these limits such that the combination $b\sqrt{\beta} \equiv \lambda$ is finite, then (4.14) becomes

$$\phi = \pi(1 + \frac{\lambda x}{\sqrt{1 + \lambda^2 x^2}}) \tag{4.15}$$

which is precisely the kink field (4.2) obtained from instantons in $\mathbb{R}^2$. As mentioned earlier the energy of this approximation is only around 1% greater than the exact solution, even though it decays only power-like as compared to the exponential decay of the exact solution (2.11). Note that in some sense this approximation obtained from instantons in $\mathbb{R}^2$ is the worst possible limit of the T^2 instanton construction. This is because the decay behaviour of (4.14) is

$$\phi(x \to -\infty) \sim \frac{4\pi}{\beta} \exp(2bx) \tag{4.16}$$

so that the approximation has an exponential decay (as does the exact solution) provided $b \neq 0$ and β is finite. It is only in the $\tau \to \infty$ case that the exponential decay is lost and replaced by a power-like decay. An indication of the parameter values for which the approximation (4.14) has minimum energy (we are using this criterion to define the 'best fit') can be obtained by comparing the decay behaviour (4.16) with that of the exact solution;

$$\phi(x \to -\infty) \sim 4 \exp(x). \tag{4.17}$$

This suggests parameter values of $b = \frac{1}{2}$ and $\beta = \pi$. Numerically minimizing the energy of (4.14) we find the parameter values $b = 0.589$ and $\beta = 1.86$, which are reasonably close to those required to optimize the decay behaviour. At these parameter values the energy of the approximate kink is $\mathcal{E} = 1.000216$ which is only $\frac{1}{50}$% above the exact value $\mathcal{E} = 1$. This approximation is therefore far superior to the approximation (4.15). It is also a slightly better approximation than any of those obtained in reference [9] from instantons in $\mathbb{R}^d$ for arbitrary positive integer d. Moreover all the approximations derived there had a power-like decay.

Having discussed the case $L \to 0$, where we have shown that the instanton generated approximation with parameters $a = 1$ and $m \to 0$ becomes exact, and the case $L \to \infty$ where we found the approximate kink with parameters $a \to \infty$ and $m \to 1$ has energy only $\frac{1}{50}$% above the true value, it is tempting to conclude that the case of general L interpolates between these two results.

This in fact proves to be correct, as can be verified by numerically integrating the holonomy formula (4.10). The results are that as L increases from 0 to ∞ the parameter values which minimize the energy of the kink approximation increase monotonically; in the case of a it is from 1 to ∞ and in the case of m it is from 0 to 1. The percentage excess energy of the approximate kink over the true solution also increases monotonically from 0% to $\frac{1}{50}$%. Note that the variation of m over its whole range is a very clear indication that the shape of the torus plays a crucial role in the accuracy of the approximation. As L increases from 0 to ∞ the shape of the torus, as measured by the ratio of the length of its sides $\frac{L}{\tau}$, also increases from 0 to ∞.

5 Conclusion

We have shown how the construction of approximate solitons from instanton holonomies may also be applied to the case of a soliton crystal. A detailed analysis shows that the excess energy of the periodic sine-Gordon kink obtained from an instanton on the torus is around $\frac{1}{50}$%, or less, for all periods. This is much smaller than the corresponding excess energy (of the order of 1%) of the kink on the infinite line obtained from instantons in the plane. A crucial ingredient in achieving this accuracy is the ability to vary the length of the torus in the euclidean time direction.

The results described here are encouraging for the use of an instanton construction of the Skyrme crystal. Particularly the fact that for the situation discussed in this paper the accuracy of the approximation is much better in the crystal case than in the previously considered infinite line example. Some preliminary results have been obtained for the Skyrme crystal and these will be described elsewhere [10].

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